R. Fuchs' problem of the Painlevé equations from the first to the fifth

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1 Introduction

About one hundred years have passed since R. Fuchs showed that the sixth Painlevé equation is represented as a monodromy preserving deformation [3]. Garnier showed that every type of the Painlevé equations is also represented as a monodromy preserving deformation [5]. Although the monodromy data is preserved by the deformations, it is difficult to calculate the monodromy data in general.

Riemann calculated the monodromy group for the Euler-Gauss hypergeometric equation. Moreover, monodromy groups of the Euler-Gauss hypergeometric equation are the polyhedron groups when it has algebraic solutions [19]. Schwarz' fifteen algebraic solutions are obtained from simple hypergeometric equations by rational transformations of the independent variable. We will study an analogue of Schwarz' solutions in the Painlevé case.

In this paper we call a linear equation whose isomonodromic deformation gives a Painlevé equation as *the Painlevé type*. For the linear equation of the Painlevé type, we can calculate the monodromy data explicitly if we substitute particular solutions of the Painlevé equations. Historically, the first

example of such solutions is Picard's solutions of the sixth Painlevé equation [18]. R. Fuchs calculated the monodromy group of the linear equation corresponding to Picard's solutions [4]. It seems that R. Fuchs' paper [4], whose title is the same as the famous paper [3], has been forgotten for long years. Recently Mazzocco found again his result independently but his paper was not referred in her paper [15]. In [4], R. Fuchs proposed the following problem:

> **R. Fuchs' Problem** Let $y(t)$ be an algebraic solution $y(t)$ of a Painlevé equation. Find a suitable transformation $x = x(z,t)$ such that the corresponding linear differential equation
>
> $$\frac{d^2v}{dz^2} = Q(t, y(t), z)v$$
>
> is changed to the form without the deformation parameter t:
>
> $$\frac{d^2u}{dx^2} = \tilde{Q}(x)u.$$
>
> Here $v = \sqrt{dz/dx}\ u$. (See the lemma 5.3.)

Picard's solution is algebraic if it corresponds to rational points of elliptic curves. For three, four and six divided points, the genus of algebraic Picard's solutions is zero. R. Fuchs showed that for algebraic Picard's solutions whose genus are zero the corresponding linear equations are reduced the Euler-Gauss hypergeometric equations by suitable rational transformations.

If R. Fuchs' problem is true, algebraic solutions of the Painlevé equations can be considered as a kind of a generalization of Schwarz' solutions. Schwarz' solutions are constructed by rational transformations which change hypergeometric equations to hypergeometric equations. Algebraic solutions of the Painlevé equations are constructed by rational transformations which change hypergeometric equations to linear equations of the Painlevé type and we can calculate the monodromy data of the linear equation explicitly.

Recently, Kitaev, jointing with Andreev, constructed many algebraic solutions of the sixth Painlevé equation, which include known ones and new ones, by rational transformations of the hypergeometric equations [1], [14]. At least now, we do not know whether R. Fuchs' problem is true or not for the sixth Painlevé equation. We do not have negative example for R. Fuchs' problem. Kitaev's transformation is a generalization of classical work by

Goursat [7]. Goursat found many rational transformations which keep the hypergeometric equations. His list is incomplete and Vidūnas made a complete list of rational transformations [21].

In this paper we will study a confluent version of [1] and we show that R, Fuchs' problem is true for the Painlevé equation from the first to the fifth. We will classify all rational transformations which change the confluent hypergeometric equations to linear equations of the Painlevé type from the first to the fifth. Compared with the sixth Painlevé equation, we obtain less transformations since the Poincaré rank of irregular singularities of the linear equations of the Painlevé type is up to three. We have up to sixth order rational functions which change the confluent hypergeometric equations to linear equations of the Painlevé type.

We show such rational transformations correspond to almost all of algebraic solutions of the Painlevé equations from the first to fifth up to the Bäcklund transformations. The cases of the degenerate fifth Painlevé equation and the Laguerre type solution of the fifth Painlevé equation are exceptional. We need exponential type transformations since the monodromy data is decomposable.

The parabolic cylinder equation (the Weber equation), the Bessel equation and the Airy equation are reduced to the confluent hypergeometric equations by rational transformations of the independent variable (see (4), (11) and (12)). These classical formula can be considered as confluent version of the Goursat transformations. Our result is an analogue of such classical formula in the Painlevé analysis.

Moreover, we obtain four non-algebraic solutions by rational transformations from the confluent hypergeometric equations. They are called as symmetric solutions of the Painlevé equations which are fixed points of simple transformations of the Painlevé equations. The symmetric solutions are not classical solutions in Umemura's sense, but it is a kind of generalization of classical solutions.

In the section 2, we will review the Painlevé equations and their special solutions. In the section 3, we will review confluent hypergeometric equations since we treat degenerate confluent hypergeometric equations. In the section 4, we will give a list of linear equations which correspond to the Painlevé equations. In the section 5, we will show that R. Fuchs' problem is true for the Painlevé equations from the first to the fifth.

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2 Remarks on the Painlevé equations

In order to fix the notation, we will review the Painlevé equations.

2.1 Remarks on P2 and P34

The thirty-fourth Painlevé equation P34(a)

$$y'' = \frac{(y')^2}{2y} + 2y^2 - ty - \frac{a}{2y}$$

appeared in Gambier's list of second order nonlinear equations without movable singularities [6]. It is known that P34 is equivalent to the second Painlevé equation P2(α)

$$y'' = 2y^3 + ty + \alpha.$$

P2(α) can be written in the Hamiltonian form

$$\mathcal{H}_{II}: \begin{cases} q' = -q^2 + p - \frac{t}{2}, \\ p' = 2pq + \left(\alpha + \frac{1}{2}\right), \end{cases} \tag{1}$$

with the Hamiltonian

$$H_{II} = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q.$$

If we eliminate p from (1), we obtain P2(α). If we eliminate q from (1), we obtain the thirty-fourth Painlevé equation P34($(\alpha+1/2)^2$). Therefore P2(α) and P34($(\alpha + 1/2)^2$) are equivalent as nonlinear differential equations, but we will distinguish these two equations from the isomonodromic viewpoint.

2.2 Remarks on P3 and P5

In the following we distinguish the three types of the third Painlevé equation P3($\alpha, \beta, \gamma, \delta$):

$$y'' = \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y},$$

because they have different forms of isomonodromic deformations:

- $D_8^{(1)}$ if $\alpha \neq 0, \beta \neq 0,\ \gamma = 0, \delta = 0$,

- $D_7^{(1)}$ if $\delta = 0, \beta \neq 0$ or $\gamma = 0, \alpha \neq 0$,
- $D_6^{(1)}$ if $\gamma\delta \neq 0$.

In the case $\beta = 0, \delta = 0$ (or $\alpha = 0, \gamma = 0$), the third Painlevé equation is a quadrature, and we exclude this case from the Painlevé family. $D_j^{(1)} (j = 6, 7, 8)$ mean the affine Dynkin diagrams corresponding to Okamoto's initial value spaces. In this paper we omit the upper index (1) for simplicity and denote D_6, D_7, D_8. For details, see [16]. By suitable scale transformations $t \to ct, y \to dt$, we can fix $\gamma = 4, \delta = -4$ for D_6. and $\gamma = 2$ for D_7. We will use another form of the third Painlevé equation P3$'(\alpha, \beta, \gamma, \delta)$

$$q'' = \frac{1}{q}q'^2 - \frac{q'}{x} + \frac{q^2(\alpha+\beta)}{4x^2} + \frac{\gamma}{4x} + \frac{\delta}{4q},$$

since P3$'$ is more sympathetic to isomonodromic deformations than P3. We can change P3 to P3$'$ by $x = t^2, ty = q$.

For the fifth equations P5$(\alpha, \beta, \gamma, \delta)$

$$\begin{aligned} y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) \\ &\quad + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}, \end{aligned}$$

we assume that $\delta \neq 0$. When $\delta = 0, \gamma \neq 0$, the fifth equation is equivalent to the third equation of the D_6 type [8]. We denote deg-P5$(\alpha, \beta, \gamma, 0)$ if $\delta = 0$. By a suitable scale transformation $t \to ct$, we can fix $\delta = -1/2$ for P5 and $\gamma = -2$ for deg-P5. Let q is a solution of P3$'(4(\alpha_1 - \beta_1), -4(\alpha_1 + \beta_1 - 1), 4, \ -4)$. Then

$$y = \frac{tq' - q^2 - (\alpha_1 + \beta_1)q - t}{tq' + q^2 - (\alpha_1 + \beta_1)q - t}$$

is a solution of deg-P5$(\alpha_1^2/2, -\beta_1^2/2, -2, 0)$.

P5$(\alpha, \beta, 0, 0)$ is quadrature and we exclude this case from the Painlevé family.

2.3 Special solutions from P1 to P5

We will review special solutions of the Painlevé equations. Although generic solutions of the Painlevé equations are transcendental, there exist some special solutions which reduce to known functions. Umemura defined a class of

‘known functions’, which are called *classical solutions* [20]. He also gave a method how to classify classical solutions of the Painlevé equations.

There exist two types of classical solutions of Painlevé equations, one is algebraic and the second is the Riccati type. Up to now, all of classical solutions are classified except algebraic solutions of the sixth Painlevé equations. We will list up all of classical solutions of P1 to P5.

Theorem 2.1. *1) All solutions of P1 are transcendental.*

2) P2(0*) has a rational solution* $y = 0$. *P2(*$-1/2$*) has a Riccati type solution* $y = -u'/u$. *Here* u *is any solution of the Airy equation* $u'' + tu/2 = 0$.

2) P34($(\alpha + 1/2)^2$*) is equivalent to P2(*α*). P34(*$1/4$*) has a rational solution* $y = t/2$. *P34(*1*) has the Riccati type solutions.*

4) P4($0, -2/9$*) has a rational solution* $y = -2t/3$. *P4(*$1 - s, -2s^2$*) has a Riccati type solution* $y = -u'/u$. *Here* u *is any solution of the Hermite-Weber equation* $u'' + 2tu' + 2su = 0$. *If* $s = 1$, *P4(*$0, -2$*) has a rational solution* $y = -2t$, *which reduces to the Hermite polynomial.*

5) P3'(D_6*)(*$a, -a, 4, -4$*) has an algebraic solution* $y = -\sqrt{t}$. *P3'(*D_6*)(*$4h, 4(h+1), 4, -4$*) has a Riccati type solution* $y = u'/u$. *Here* u *is any solution of* $tu'' + (2h + 1)u' - 4tu = 0$.

6) P3'(D_7*)(*$\alpha, \beta, \gamma, 0$*) does not have the Riccati type solution. P3'(*D_7*)(*$0, -2, 2, 0$*) has an algebraic solution* $y = t^{1/3}$.

7) P3'(D_8*)(*$\alpha, \beta, 0, 0$*) does not have the Riccati type solution. P3'(*D_8*) (*$8h, -8h, 0, 0$*) has an algebraic solution* $y = -\sqrt{t}$.

8) P5($a, -a, 0, \delta$*) has a rational solution* $y = -1$. *P5(*$(\kappa_0+s)^2/2, -\kappa_0^2/2, -(s+1), -1/2$*) has the Riccati type solutions* $y = -tu'/(\kappa_0 + s)u$. *Here* u *is any solution of* $t^2u'' + t(t - s - 2\kappa_0 + 1)u' + \kappa_0(\kappa_0 + s)u2 = 0$. *If* $\kappa_0 = 1$, *P5(*$(s+1)^2/2, -1/2, -(s+1), -1/2$*) has a rational solution* $y = t/(s+1)+1$, *which reduces to the Laguerre polynomial.*

9) deg-P5($\alpha_1^2/2, -\beta_1^2/2, -2, 0$*) is equivalent to P3(*$D_6$*)(*$4(\alpha_1-\beta_1), -4(\alpha_1+\beta_1-1), 4,\ -4$*). deg-P5(*$h^2/2, -8, -2, 0$*) has an algebraic solution* $y = 1 + 2\sqrt{t}/h$. *deg-P5(*$\alpha, 0, \gamma, 0$*) has the Riccati type solutions.*

10) All of classical solutions of P1 to P5 are equivalent to the above solutions up to the Bäcklund transformations.

2.4 Symmetric solutions for P1, P2, P34 and P5

We will review symmetric solutions [13], [10]. The first, second, thirty-fourth and fourth Painlevé equations

$$\begin{array}{ll} \mathrm{P1} & y'' = 6y^2 + t, \\ \mathrm{P2}(\alpha) & y'' = 2y^3 + ty + \alpha, \\ \mathrm{P34}(a) & y'' = \dfrac{(y')^2}{2y} + 2y^2 - ty - \dfrac{a}{2y}, \\ \mathrm{P4}(\alpha,\beta) & y'' = \dfrac{1}{2y}y'^2 + \dfrac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \dfrac{\beta}{y}, \end{array}$$

have a simple symmetry:

$$\begin{array}{ll} \mathrm{P1} & y \to \zeta^3 y, \quad t \to \zeta t, \quad (\zeta^5 = 1) \\ \mathrm{P2} & y \to \omega y, \quad t \to \omega^2 t, \quad (\omega^3 = 1) \\ \mathrm{P34} & y \to \omega y, \quad t \to \omega t, \quad (\omega^3 = 1) \\ \mathrm{P4} & y \to -y, \quad t \to -t, \end{array}$$

There exist symmetric solutions which are invariant under the action of the cyclic group. The symmetric solutions are studied by Kitaev [13] for P1 and P2 and by Kaneko [10] for P4. Since these symmetric solutions exist for any parameter of the Painlevé equations, they are not algebraic for generic parameters.

Theorem 2.2. *1) For P1, we have two symmetric solutions*

$$\begin{aligned} y &= \frac{1}{6}t^3 + \frac{1}{336}t^8 + \frac{1}{26208}t^{13} + \frac{95}{224550144}t^{18} + \cdots, \\ y &= t^{-2} - \frac{1}{6}t^3 + \frac{1}{264}t^8 - \frac{1}{19008}t^{13} + \cdots. \end{aligned}$$

2) For P2(α), we have three symmetric solutions

$$\begin{aligned} y &= \frac{\alpha}{2}t^2 + \frac{\alpha}{40}t^5 + \frac{10\alpha^3+\alpha}{40}t^8 + \cdots, \\ y &= t^{-1} - \frac{\alpha+1}{4}t^3 + \frac{(\alpha+1)(3\alpha+1)}{112}t^5 + \cdots, \\ y &= -t^{-1} - \frac{\alpha-1}{4}t^3 - \frac{(\alpha-1)(3\alpha-1)}{112}t^5 + \cdots. \end{aligned}$$

They are equivalent to each other by the Bäcklund transformations.

2) For P34(a^2), we have three symmetric solutions

$$
\begin{aligned}
y &= at + \frac{a(2a-1)}{8}t^4 + \frac{a(2a-1)(10a-3)}{560}t^7 + \cdots , \\
y &= -at + \frac{a(2a+1)}{8}t^4 - + \frac{a(2a+1)(10a+3)}{560}t^7 + \cdots , \\
y &= \frac{2}{t^2} + \frac{t}{2} - \frac{4a^2-9}{224}t^4 - \frac{4a^2-9}{5600}t^7 + \cdots .
\end{aligned}
$$

Each solution corresponds to the symmetric solutions of P2, respectively, by

$$y_{34} = y_2' + y_2^2 + \frac{t}{2}.$$

4) For P4($\alpha, -8\theta_0^2$), we have four symmetric solutions

$$
\begin{aligned}
y &= \pm 4\theta_0 \left(t - \frac{2\alpha}{3}t^3 + \frac{2}{15}(\alpha^2 + 12\theta_0^2 \pm \theta_0 + 1)t^5 + \cdots \right), \\
y &= \pm t^{-1} + \frac{2}{3}(\pm\alpha - 2)t^3 \mp \frac{2}{45}(-7\alpha^2 \pm 16\alpha + 36\theta_0^2 - 4)t^5 + \cdots .
\end{aligned}
$$

They are equivalent to each other by the Bäcklund transformations.

We remark that there are no Bäcklund transformation for P1. We may think symmetric solutions as a generalization of Umemura's classical solutions. We notice that symmetric solutions are not classical functions except for special parameters. For example, the first solution of P2(α) is transcendental for a generic parameter, but it is a rational solution $y = 0$ for $\alpha = 0$. It is rational if and only if $\alpha/3$ is an integer. The symmetric solutions exist for any parameters and classical solutions exist only for special parameters.

3 Confluent hypergeometric equation

In this section we review Kummer's confluent hypergeometric equation. It is known that there exist two standard forms, Kummer's form and Whittaker's form for the confluent hypergeometric equation. At the first we will use Kummer's standard form.

We review irregular singularities to fix our notations. For a rational function $a(x)$, we set

$$\mathrm{ord}_{x=c}a(x) = \text{ (pole order of } a(x) \text{ at } x = c).$$

A linear differential equation

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x) = 0$$

has an irregular singularity at $x = c$ if and only if $\mathrm{ord}_{x=c}p(x) > 1$ or $\mathrm{ord}_{x=c}q(x) > 2$. The Poincaré rank r of the irregular singularity $x = c$ is

$$r = \max\{\mathrm{ord}_{x=c}p(x), 1/2 \cdot \mathrm{ord}_{x=c}q(x)\} - 1.$$

We think that a regular singularity has the Poincaré rank $r = 0$.

Kummer's confluent hypergeometric equation is

$$x\frac{d^2u}{dx^2} + (c - x)\frac{du}{dx} - au = 0, \tag{2}$$

which has a regular singularity at $x = 0$ and an irregular singularity with the Poincaré rank 1 at $x = \infty$. We set $x \to \varepsilon x, a \to 1/\varepsilon$ in (2) and take the limit $\varepsilon \to 0$. Then we get a degenerate confluent hypergeometric equation (the confluent hypergeometric limit equation)

$$x\frac{d^2u}{dx^2} + c\frac{du}{dx} - u = 0, \tag{3}$$

which has a regular singularity at $x = 0$ and an irregular singularity with the Poincaré rank 1/2 at $x = \infty$. The solution of (3) is

$$y = C\ {}_0F_1(c; x) + D\ x^{1-c}{}_0F_1(2 - c; x).$$

(3) is reduced to (2) by Kummer's second formula

$$ {}_0F_1\left(c; x^2/16\right) = e^{-x/2}\ {}_1F_1(c - 1/2, 2c - 1; x).$$

It is also related to the Bessel function

$$ {}_0F_1\left(c; x^2/16\right) = \Gamma(c)(-ix/4)^{1-c}J_{c-1}(-ix/2). \tag{4}$$

Later we will use SL-type equations in the section4.2. SL-type of the confluent hypergeometric equation is called the Whittaker equation:

$$W_{k,m}: \quad \frac{d^2u}{dx^2} = \left(\frac{1}{4} - \frac{k}{x} + \frac{m^2 - \frac{1}{4}}{x^2}\right)u = 0, \tag{5}$$

$$DW_m: \quad \frac{d^2u}{dx^2} = \left(\frac{1}{x} + \frac{m^2 - \frac{1}{4}}{x^2}\right)u = 0. \tag{6}$$

In $W_{k,m}$, the parameters k, m correspond to $k = c/2 - a, \quad m = (c - 1)/2$ in (2). In DW_m, the parameter m corresponds to $m = (c - 1)/2$ in (3).

4 Linear equations of the Painlevé type

We call a linear equation of the second order whose isomonodromic deformation gives a Painlevé equation as *the Painlevé type*. In this section we will list up all of linear equations of the Painlevé type. Since a linear equation of the second order is equivalent to a 2×2 system of linear equations of the first order, we use a single equation. It is easy to rewrite as a 2×2 system.

4.1 Singularity type

Linear differential equations of the Painlevé type have the following types of singular points.

P6	$(0)^4$
P5	$(0)^2(1)$
P3(D_6)	$(1)^2$ or $(0)^2(1/2)$
P3(D_7)	$(1)(1/2)$
P3(D_8)	$(0)(2)$
P4	$(0)(2)$
P34	$(0)(3/2)$
P2	(3)
P1	$(5/2)$

Here $(m)^k$ means k singular points with the Poincaré rank m, and (0) means a regular singularity. The Painlevé third equation has two type of singularities. $(1)^2$ is a standard one, and $(0)^2(1/2)$ is deg-P5.

Two different types of linear equations are used for the isomonodromic deformation of the Painlevé second equation. One is Garnier's form [5], which is the same one used by Okamoto [17] and Miwa-Jimbo [9], The second one is Flaschka-Newell's form [2], which is is equivalent to the type $(0)(3/2)$ by a rational transformation [12]. It is natural that Flaschka-Newell's form as an isomonodromic deformation for P34. We will study Garnier's form and Flaschka-Newell's form in a succeeding paper.

4.2 List of equations of the Painlevé type

In the following, we will list up all of linear equations of the Painlevé type. We use linear equations of SL-type:

$$\frac{d^2u}{dz^2} = p(z,t)u. \tag{7}$$

Isomonodromic deformation for a linear equation of SL-type is given by the compatibility condition of the following system [17]:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= V(x,t)u, \\ \frac{\partial u}{\partial t} &= A(x,t)\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial A(x,t)}{\partial x}u. \end{aligned} \tag{8}$$

We will list up $V(x,t)$ and $A(z)$ of linear equations of Painlevé type. We corrected misprints in [17]. The compatibility condition gives the Painlevé equation, which turn out a Hamiltonian system with the Hamiltonian K_J in the following list. (q,p) are canonical coordinates and q satisfies the Painlevé equation.

Type $(5/2)$: the first Painlevé equation P1

$$V(z,t) = 4z^3 + 2tz + 2K_{\mathrm{I}} + \frac{3}{4(z-q)^2} - \frac{p}{z-q}$$

$$A(z) = \frac{1}{2}\cdot\frac{1}{z-q}$$

$$K_{\mathrm{I}} = \frac{1}{2}p^2 - 2q^3 - tq.$$

Type (3) : the second Painlevé equation P2(α)

$$V(z,t) = z^4 + tz^2 + 2\alpha z + 2K_{\mathrm{II}} + \frac{3}{4(z-q)^2} - \frac{p}{z-q}$$

$$A(z) = \frac{1}{2}\cdot\frac{1}{z-q}$$

$$K_{\mathrm{II}} = \frac{1}{2}p^2 - \frac{1}{2}q^4 - \frac{1}{2}tq^2 - \alpha q.$$

Type (1)(3/2) : the thirty-fourth Painlevé equation P34(α)

$$V(z,t) = \frac{z}{2} - \frac{t}{2} + \frac{\alpha - 1}{4z^2} - \frac{K_{\mathrm{XXXIV}}}{z} + \frac{3}{4(z-q)^2} - \frac{pq}{z(z-q)}$$

$$A(z) = -\frac{z}{z-q}$$

$$K_{\mathrm{XXXIV}} = -qp^2 + p + \frac{q^2}{2} - \frac{tq}{2} + \frac{\alpha - 1}{4q}.$$

Type (1)(2) : the fourth Painlevé equation P4(α, β)

$$V(z,t) = \frac{a_0}{z^2} + \frac{K_{\mathrm{VI}}}{2z} + a_1 + \left(\frac{z+2t}{4}\right)^2 + \frac{3}{4(z-q)^2} - \frac{pq}{z(z-q)},$$

$$A(z) = \frac{2z}{z-q},$$

$$K_{\mathrm{VI}} = 2qp^2 - 2p - \frac{a_0}{q} - 2a_1 q - 2q\left(\frac{q+2t}{4}\right)^2,$$

$$a_0 = -\frac{\beta}{8} - \frac{1}{4},\ a_1 = -\frac{\alpha}{4}.$$

Type $(1)^2$: the third Painlevé equation P3$'$($\alpha, \beta, \gamma, \delta$)

$$V(z,t) = \frac{a_0 t^2}{z^4} + \frac{a_0' t}{z^3} - \frac{tK_{\mathrm{III}}'}{z^2} + \frac{a_\infty'}{z} + a_\infty + \frac{3}{4(z-q)^2} - \frac{pq}{z(z-q)},$$

$$A(z) = \frac{qz}{t(z-q)},$$

$$tK_{\mathrm{III}}' = q^2p^2 - qp - \frac{a_0 t^2}{q^2} - \frac{a_0' t}{q} - a_\infty' q - a_\infty q^2,$$

$$a_0 = -\frac{\delta}{16},\ a_0' = -\frac{\beta}{8},\ a_\infty = \frac{\gamma}{16},\ a_\infty' = \frac{\alpha}{8}.$$

Type $(0)^2(1)$: the fifth Painlevé equation P5($\alpha, \beta, \gamma, \delta$)

$$V(z,t) = \frac{a_1 t^2}{(z-1)^4} + \frac{K_{\mathrm{V}} t}{(z-1)^2 z} + \frac{a_2 t}{(z-1)^3} - \frac{p(q-1)q}{z(z-1)(z-q)} + \frac{a_\infty}{(z-1)^2} + \frac{a_0}{z^2} + \frac{3}{4(z-q)^2}$$

$$A(z) = \frac{q-1}{t} \cdot \frac{z(z-1)}{z-q}$$

$$tK_{\mathrm{V}} = q(q-1)^2 \left[-\frac{a_1 t^2}{(q-1)^4} - \frac{a_2 t}{(q-1)^3} + p^2 - \left(\frac{1}{q} + \frac{1}{q-1} \right) p - \frac{a_\infty}{(q-1)^2} - \frac{a_0}{q^2} \right]$$

$$a_0 = -\frac{\beta}{2} - \frac{1}{4},\ a_1 = -\frac{\delta}{2},\ a_2 = -\frac{\gamma}{2},\ a_\infty = \frac{1}{2}(\alpha+\beta) - \frac{3}{4}$$

If $\gamma = 0$ or $\delta = 0$ for P3, the type of the linear equation is (1)(1/2). We will take $\delta = 0$ as a standard form, which reduces to P3(D_7)($\alpha, \beta, \gamma, 0$). If $\gamma = 0$ and $\delta = 0$ for P3, the type of the linear equation is $(1/2)^2$, which reduces to P3(D_8)($\alpha, \beta, 0, 0$). If $\delta = 0$ for P5, the type of the linear equation is $(0)^2(1/2)$, which reduces to deg-P5($\alpha, \beta, \gamma, 0$). We omit P6 since we treat the cases of irregular singularities in this paper.

5 Rational transform of $W_{k,m}$ and DW_m

In this section, we will classify all of rational transformations of independent variables, which change $W_{k,m}$ or DW_m into linear differential equations of the Painlevé type.

The following simple lemma is a key of classification.

Lemma 5.1. *For a linear differential equation*

$$\frac{d^2u}{dx^2} = p(x)u(x), \tag{9}$$

we take a new independent variable z defined by a rational transform $x = x(z)$. If $x = c$ is a regular singularity, $z_c = x^{-1}(c)$ is also a regular singularity of the transformed equation. When z_c is a n-th branched point and the distance of the two exponent at $x = c$ is $1/n$, $z = z_c$ is an apparent singularity and $z = z_c$ can be reduced to a regular point by a suitable change $y = v(z)u$.

If $x = c$ is an irregular singularity of the Poincaré rank m and z_c is a n-th branched point, $z = z_c$ is an irregular singularity of the Poincaré rank mn.

The proof is obvious. Since we will start from the confluent hypergeometric equations, we take a care of $x = 0, \infty$ as branched points. For $x = x(z)$, if $x^{-1}(0)$ are branch points with $\mu_1, \mu_2, \cdots$ order and $x^{-1}(\infty)$ are branch points

with $\nu_1, \nu_2, \cdots$ order, we call $x(z)$ as a type $(\mu_1 + \mu_2 + \cdots | \nu_1 + \nu_2 + \cdots)$. We allow μ_j or ν_j equals to one. $\sum \mu_j = \sum \nu_j$ is the order of the rational function $x(z)$.

Theorem 5.2. *By a rational transform $x = x(z)$, $W_{k,m}$ or DW_m changes to a linear equation of the Painlevé type or a confluent hypergeometric equation if and only if one of the following cases occur.*

1) Double cover

$W_{k,m}$	$(2\|2)$	$(0)(2)$	*P4-sym*
$W_{k,1/4}$	$(2\|2)$	(2)	*Weber*
$W_{k,1/4}$	$(2\|1+1)$	$(1)^2$	*D6-alg*
$W_{0,1/2}$	$(1+1\|2)$	$(0)(2)$	*P4-Her*
DW_m	$(2\|2)$	$(0)(1)$	*Kummer*
DW_m	$(1+1\|2)$	$(1)^2(2)$	*P5-rat*
$DW_{1/4}$	$(2\|1+1)$	$(1/2)^2$	*D8-alg*

2) Cubic cover

$W_{k,1/3}$	$(3\|3)$	(3)	*P2-sym*
DW_m	$(3\|3)$	$(1)(3/2)$	*P34-sym*
$DW_{1/6}$	$(3\|3)$	$(3/2)$	*Airy*
$DW_{1/4}$	$(2+1\|3)$	$(0)(3/2)$	*P34-rat*
$DW_{1/6}$	$(3\|2+1)$	$(1)(1/2)$	*D7-alg*

3) Quartic cover

$DW_{1/6}$	$(3+1\|4)$	(3)	*P4-rat*

4) Quintic cover

$DW_{1/5}$	$(5\|5)$	$(5/2)$	*P1-sym*
$DW_{1/10}$	$(5\|5)$	$(5/2)$	*P1-sym*

5) Sextic cover

$DW_{1/6}$	$(3+3\|6)$	(3)	*P2-rat*

Here the first column is the starting linear equation. The second column is the type of a rational transform. The third column is the singularity type of of the transformed linear equation. The fourth column is the solution of the Painlevé equation.

Proof. The Poincare rank of irregular singularities of equations of the Painlevé type is at most three. Therefore we can take up to a cubic cover of $W_{k,m}$ and

we can take up to a sextic cover of DW_m. In each cases, we can classify the covering map directly. $\square$

The other types of confluent equations, such as the Weber (parabolic cylinder) equation and the Airy equation, appear in the table. Therefore covering of such equations are also included in the covering of $W_{k,m}$ or DW_m. We consider the Bessel equation as a special case of $W_{0,m}$. We will explain each case in the next subsection. The theorem 5.2 shows that it is necessary to distinguish P2 and P34 from the isomonodromic viewpoint.

5.1 Exact form

In this section we will write down algebraic solutions obtained by rational transformations from $W_{k,m}$ or DW_m explicitly. The following lemma is a key to calculate exact transformations.

Lemma 5.3. *For the equation*

$$\frac{d^2u}{dx^2} = Q(x)u,$$

we set

$$x = x(z),\ \ u(x) = \sqrt{\frac{dx}{dz}}v(z).$$

Then v satisfies

$$\frac{d^2v}{dz^2} = \left(Q(x(z))(x'(z))^2 - \frac{1}{2}\{z,x\}\right)v. \tag{10}$$

Here $\{z,x\}$ is the Schwarzian derivative

$$\{z,x\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right).$$

We will take $Q(x)$ as (5) or (6). Then we calculate $V(z,t)$ in the left hand side of (10) in each case of the theorem 5.2. We should take a suitable coefficients of the rational function $x = x(z)$ to coincide with the equation of the Painlevé type. In the following (q,p) is the canonical coordinates in the section 4.2.

5.1.1 Double cover

1) P4-sym $(2|2)$
For $W_{k,m}$, we set $x = z^2/4$. Then

$$V(z,t) = -k + \frac{16m^2-1}{4z^2} + \frac{z^2}{16},$$

which is the case $t=0$ for the symmetric solution $q(t) = 2(4m+1)t + O(t^3)$ of P4$(4k, -2(4m+1)^2)$.

2) Weber $(2|2)$
This is a special case of P4-sym. For $W_{k,1/4}$, we set $x = z^2/2$. Then

$$V(z,t) = \frac{z^2}{4} - 2k,$$

which is the parabolic cylinder equation for $D_{2k-1/2}(z)$. This shows the well-known formula

$$D_{2k-1/2}(z) = 2^k z^{-1/2} W_{k,-1/4}\left(\frac{z^2}{2}\right). \tag{11}$$

3) D_6-alg $(2|1+1)$
For $W_{k,1/4}$, we set $x = (z-\sqrt{t})^2/z$. Then

$$V(z,t) = \frac{1}{4} + \frac{t^2}{4z^2} - \frac{kt}{z^3} - \frac{8t+32k\sqrt{t}+3}{16z^2} - \frac{k}{z} + \frac{3}{4(z+\sqrt{t})^2} - \frac{3}{4z(z+\sqrt{t})},$$

which gives the algebraic solution $q(t) = -\sqrt{t}$ of P3$(-8k, 8k, 4, -4)$.

4) P4-Her $(1+1|2)$
For $W_{0,1/2}$, we set $x = z(z+4t)/4$. Then

$$V(z,t) = \left(\frac{z+2t}{4}\right)^2 + \frac{4}{3(z+2t)},$$

which gives the rational solution $q(t) = -2t$ of P4$(0,-2)$.

5) Kummer's second formula $(2|2)$
For DW_m, we set $x = z^2/16$. Then

$$V(z,t) = \frac{1}{4} + \frac{16m^2-1}{4z^2},$$

which is $W_{0,2m}$. This is Kummer's second formula.

6) P5-rat $(1+1|2)$
For DW_m, we set $x = ht^2 z/4(z-1)^2$. Then

$$V(z,t) = \frac{ht^2}{(z-1)^4} + \frac{16m^2 - 1 + ht^2}{4z(z-1)^2} - \frac{3}{4z} + \frac{4m^2-1}{4z^2} + \frac{3(z+2)}{4(z+1)^2},$$

which gives the rational solution $q(t) = -1$ of $\mathrm{P5}(2m^2, -2m^2, 0, -2h)$.

7) D_8-alg $(2|1+1)$
For $DW_{1/4}$, we set $x = h(z-\sqrt{t})^2/z$. Then

$$V(z,t) = \frac{ht}{z^3} + \frac{32h\sqrt{t} - 3}{16z^2} + \frac{h}{z} + \frac{3}{4(z+\sqrt{t})^2} - \frac{3}{4z(z+\sqrt{t})},$$

which gives the algebraic solution $q(t) = -\sqrt{t}$ of $\mathrm{P3}(8h, -8h, 0, 0)$.

5.1.2 Cubic cover

1) P2-sym $(3|3)$
For $W_{k,1/3}$, we set $x = 2z^3/3$. Then

$$V(z,t) = z^4 - 6kz + \frac{3}{4z^2},$$

which is the symmetric solution $q(0) = 0, p(0) = 0$ of $\mathrm{P2}(-3k)$.

2) P34-sym $(3|3)$
For DW_m, we set $x = z^3/18$. Then

$$V(z,t) = \frac{z}{2} + \frac{36m^2-1}{4z^2},$$

which is the symmetric solution $q(0) = 0$ of $\mathrm{P34}(3(12m^2-1))$. We have two symmetric solutions with $q(0) = 0$, and both give the same equation when $t = 0$.

3) Airy $(3|3)$
This is a special case of P34-sym. For $DW_{1/6}$, we set $x = z^3/9$. Then

$$V(z,t) = z,$$

which is the Airy equation. It is know that

$$\mathrm{Ai}(x) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})}{}_0F_1\left(\frac{2}{3};\frac{z^3}{9}\right) - \frac{x}{3^{1/3}\Gamma(\frac{1}{3})}{}_0F_1\left(\frac{4}{3};\frac{z^3}{9}\right). \tag{12}$$

4) P34-rat $(2+1|3)$
For $DW_{1/4}$, we set $x = z(z-3t/2)^3/18$. Then

$$V(z,t) = \frac{z}{2} - \frac{t}{2} + \frac{t^3/4+1}{2tz} - \frac{3}{16z^2} + \frac{3}{4(z-t/2)^2} - \frac{1}{2t(z-t/2)},$$

which gives the rational solution $q(t) = t/2$ of P34(1/4).

5) D_7-alg $(3|2+1)$
For $DW_{1/6}$, we set $x = \left(z+2t^{1/3}\right)^3/32z$. Then

$$V(z,t) = \frac{t}{4z^3} - \frac{16+27t^{2/3}}{72z^2} + \frac{2}{3t^{1/3}z} + \frac{1}{8} + \frac{3}{4(z-t^{1/3})^2} - \frac{2}{3t^{1/3}(z-t^{1/3})},$$

which is the algebraic solution $q(t) = t^{1/3}$ of P3$(0,-2,2,0)$.

5.1.3 Quartic cover

If the covering degree is more than three, we start only from DW_m. If a quartic covering map $x = x(z)$ splits to $x = (\bar{x}(z))^2$, the covering through $W_{0,2m}$, which is Kummer's case 5.1.4. Such case occurs in case that all of μ_j, ν_j are even. $(4|4), (4|2{+}2), (2{+}2|4)$ corresponds to $(2|2), (2|1{+}1), (1{+}1|2)$, respectively.

1) P4-rat $(3+1|4)$
For $DW_{1/6}$, we set $x = \frac{1}{256}z\left(z+8t/3\right)^3$. Then

$$V(z,t) = -\frac{2}{9z^2} + \frac{2t^3}{27z} + \left(\frac{z+2t}{4}\right)^2 + \frac{27}{4(3z+2t)^2} - \frac{1}{z(3z+2t)}$$

which gives the rational solution $q(t) = -2t/3$ of P4$(0,-2/9)$.

5.1.4 Quintic cover

1) P1-sym $(3+1|4)$
For $DW_{1/5}$, we set $x = 4z^5/25$. Then

$$V(z,t) = \frac{3}{4z^2} + 4z^3,$$

which is the symmetric solution $y(0) = 0, y'(0) = 0, t = 0$ of P1.

For $DW_{1/10}$, we set $x = 4z^5/25$. Then

$$V(z,t) = 4z^3,$$

which is the symmetric solution $y(t) = 1/t^2 + \cdots$ and $t = 0$ of P1. If we substitute $y(t) = 1/t^2 + \cdots$ in $Q(z,t)$, $Q(z,t)$ may have a pole $t = 0$ but this pole is apparent.

5.1.5 Sextic cover

As the same as the quartic cover, we omit the case $(6|6), (4+2|6)$.

1) P2-rat $(3+3|6)$
For $DW_{1/6}$, we set $x = \frac{1}{36}(z^2+t)^3$. Then

$$V(z,t) = \frac{3}{4z^2} + tz^2 + z^4,$$

which is the rational solution $q = 0$ of P2(0).

5.2 R. Fuchs' Problem

Compared with the theorem 2.1 and the theorem 5.2, we obtain all of algebraic solutions and symmetric solutions except algebraic solutions of deg-P5 and the Laguerre solutions of P5. These two solutions are not obtained from $W_{k,m}$ or DW_m by rational transformations, but it is obtained by exponential type transformations.

5.2.1 Split case

1) For deg-P5-alg, we start from

$$\frac{d^2u}{dx^2} = \frac{h^2-1}{4x^2}u.$$

We set $x = e^{4\sqrt{tz/(z-1)}/h}(\sqrt{z}+\sqrt{z-1})/(\sqrt{z}-\sqrt{z-1})$. Then

$$\begin{aligned} V(z,t) = {} & \frac{t}{(z-1)^3} - \frac{4h^2-13}{16(z-1)^2} - \frac{3}{16z^2} - \frac{2(h+2\sqrt{t})^2-5}{8z(z-1)^2} \\ & + \frac{3}{4(z-2\sqrt{t}/h-1)^2} - \frac{3+8\sqrt{t}/h}{4z(z-1)(z-1-2\sqrt{t}/h)}, \end{aligned}$$

which gives an algebraic solution $y = 1+2\sqrt{t}/h$ for P5$(h^2/2, -8, -2, 0)$.

2) For P5-Lag, we start from

$$\frac{d^2u}{dx^2} = \frac{h^2-1}{4x^2}u.$$

We set $x = e^{t/(h(z-1))}(z-1)$. Then

$$V(z,t) = \frac{t^2}{4(z-1)^4} - \frac{ht}{2(z-1)^3} + \frac{h^2/4-1}{(z-1)^2} - \frac{3}{4(z-t/h-1)^2} - \frac{ht}{(z-1)^2(z-t/h-1)},$$

which gives a rational solution $y = t/h+1$ for P5$(h^2/2, -1/2, -h, -1/2)$.

In these two cases, the monodromy group of

$$\frac{d^2u}{dz^2} = V(z,t)u(z) \tag{13}$$

is diagonal. Therefore we cannot reduce them to $W_{k,m}$ or DW_m.

5.2.2 Summary

We summarize our result:

Theorem 5.4. *By any rational transformation from confluent hypergeometric equations $W_{k,m}$ or DW_m to equations of the Painlevé type, we get algebraic solutions or symmetric solutions of P1, P2, P34, P3, P4, P5 except*

deg-P5-alg and P5-Lag. Conversely, any algebraic solution except deg-P5-alg and P5-Lag or symmetric solution from P1 to P5 is obtained by a rational transformation of $W_{k,m}$ or DW_m. For deg-P5-alg and P5-Lag, we can reduce them to differential equations with constant coefficients by exponential type transformations.

In this sense, R. Fuchs' problem is true for P1 to P5. We obtain rational solutions and symmetric solutions of P2 and P34 in different way. In [13], Kitaev studied a symmetric solution of P2, but his result is on a symmetric solution of P34 from our viewpoint since he used Flaschka-Newell's form. Symmetric solutions of P2 is studied in [11] by using Miwa-Jimbo's form.

The authors do not know R. Fuchs' problem is true or not in the case the sixth Painlevé equations. Recent work by A. V. Kitaev [14] give partial, but affirmative answers to R. Fuchs' problem. We do not know any negative examples of R. Fuchs' problem.

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